Analysis of Solution Structure in Reaction-Diffusion Equations with Discontinuous Coefficients

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1 Introduction

In this report, we investigate the following equation:

$$\begin{cases} u_t(x,t) - (K(x)u_x(x,t))_x + V(x)u(x,t) = 0 \\ u(x,0) = \delta(x-y) \end{cases}$$

where K, V are step function and y a fixed number in $\mathbb{R}$. This framework can be used to simulate fluid dynamics involving reaction and diffusion in various mediums. We employ the method of the Laplace wave train to find the solution to this equation.

2 Preliminaries

Proposition 2.1. Given two coefficient K and V. The solution of

$$\begin{cases} u_t(x,t) - Ku_{xx}(x,t) + Vu(x,t) = 0 \\ u(x,0) = \delta(x-y) \end{cases} \tag{2.1}$$

is

$$u(x,t) = \frac{e^{-Vt-\frac{(x-y)^2}{4Kt}}}{2\sqrt{K\pi t}} \tag{2.2}$$

Proposition 2.2. Denotes the Laplace transform of function $f(x,t)$ on on variable t as

$$\mathcal{L}(f)(x,s) \equiv \int_0^\infty e^{-st}f(x,t)dt,\ \mathrm{Re}(s) > 0$$

Then, the Laplace form of (2.2) on variable t is

$$\mathcal{L}(\frac{e^{-Vt-\frac{(x-y)^2}{4Kt}}}{2\sqrt{K\pi t}}) = \frac{1}{2\sqrt{K(s+V)}}e^{-\sqrt{\frac{s+V}{K}}|x-y|} \tag{2.3}$$

Proof: By applying Laplace transform on (2.1),

$$\begin{aligned} s\mathcal{L}u - K\partial_x^2\mathcal{L}u + V\mathcal{L}u - \delta(x-y) = 0 \\ (s + V - K\partial_x^2)\mathcal{L}u = \delta(x-y) \end{aligned} \tag{2.4}$$

which gives

$$\mathcal{L}u = \frac{1}{2\sqrt{K(s+V)}} e^{-\sqrt{\frac{s+V}{K}}|x-y|} \tag{2.5}$$

By (2.2) and (2.5),

$$\mathcal{L}\left(\frac{e^{-Vt-\frac{(x-y)^2}{4Kt}}}{2\sqrt{K\pi t}}\right) = \frac{1}{2\sqrt{K(s+V)}} e^{-\sqrt{\frac{s+V}{K}}|x-y|} \tag{2.6}$$

□

By apply partial differential of x at both side of (2.6), we have

$$\mathcal{L}\left(|x-y|\frac{e^{-Vt-\frac{(x-y)^2}{4Kt}}}{2\sqrt{K\pi}t^{3/2}}\right) = e^{-\sqrt{\frac{s+V}{K}}|x-y|} \tag{2.7}$$

3 Reflection and transmission coefficients around a jump

Consider the case that K and V are step functions with one jump at 0, that is,

$$\begin{cases} u_t(x,t) - (K(x)u_x)_x(x,t) + V(x)u(x,t) = 0 \\ K(x) = K_+\mathcal{H}(x) + K_-(1-\mathcal{H}(x)) \\ V(x) = V_+\mathcal{H}(x) + V_-(1-\mathcal{H}(x)) \\ u(x,0) = \delta(x-y) \end{cases} \tag{3.1}$$

where $\mathcal{H}(x)$ is Heaviside function,

$$\mathcal{H}(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases}$$

Apply the Laplace transform with respect to variable t on (3.1), we have

$$s\mathcal{L}u - (K(x)\mathcal{L}u_x)_x + V(x)\mathcal{L}u = \delta(x-y) \tag{3.2}$$

and it implied by (2.4) and (2.5) that for some coefficients $R_{--}, R_{++}, T_{-+}, T_{+-}$,

$$\mathcal{L}u = \begin{cases} \left(\frac{e^{-\sqrt{\frac{s+V_-}{K_-}}|x-y|}}{2\sqrt{K_-(s+V_-)}} + R_{--}\frac{e^{-\sqrt{\frac{s+V_-}{K_-}}|x+y|}}{2\sqrt{K_-(s+V_-)}}\right)(1-\mathcal{H}(x)) + \left(T_{-+}\frac{e^{-\sqrt{\frac{s+V_-}{K_-}}|y|-\sqrt{\frac{s+V_+}{K_+}}|x|}}{2\sqrt{K_-(s+V_-)}}\right)\mathcal{H}(x) & ,y<0 \\ \left(T_{+-}\frac{e^{-\sqrt{\frac{s+V_+}{K_+}}|y|-\sqrt{\frac{s+V_-}{K_-}}|x|}}{2\sqrt{K_+(s+V_+)}}\right)(1-\mathcal{H}(x)) + \left(\frac{e^{-\sqrt{\frac{s+V_+}{K_+}}|x-y|}}{2\sqrt{K_+(s+V_+)}} + R_{++}\frac{e^{-\sqrt{\frac{s+V_+}{K_+}}|x+y|}}{2\sqrt{K_+(s+V_+)}}\right)\mathcal{H}(x) & ,y>0 \end{cases} \tag{3.3}$$

By requiring the continuity condition,

1. $\mathcal{L}u$ is continuous at $x = 0$,

2. $\mathcal{L}u_x$ is continuous at $x = 0$,

we obtain that the reflection and transmission coefficients are

$$\begin{cases} T_{+-} &= \dfrac{2\sqrt{K_+(s+V_+)}}{\sqrt{K_-(s+V_-)}+\sqrt{K_+(s+V_+)}} \\ \\ T_{-+} &= \dfrac{2\sqrt{K_-(s+V_-)}}{\sqrt{K_-(s+V_-)}+\sqrt{K_+(s+V_+)}} \\ \\ R_{++} &= \dfrac{\sqrt{K_+(s+V_+)}-\sqrt{K_-(s+V_-)}}{\sqrt{K_-(s+V_-)}+\sqrt{K_+(s+V_+)}} \\ \\ R_{--} &= \dfrac{\sqrt{K_-(s+V_-)}-\sqrt{K_+(s+V_+)}}{\sqrt{K_-(s+V_-)}+\sqrt{K_+(s+V_+)}} \end{cases} \tag{3.4}$$

4 Some definition

Let $J = \{x_1, x_2, \ldots, x_N\}$ be the set of all of jump discontinuous of $K(x)$ or $V(x)$. We define the transmission coefficients $T^{x_j}_{-+}, T^{x_j}_{+-}$ and reflection coefficients $R^{x_j}_{--}, R^{x_j}_{++}$ across x_j as,

$$\begin{cases} T^{x_j}_{+-} = \dfrac{2\sqrt{K_+(s+V_+)}}{\sqrt{K_-(s+V_-)}+\sqrt{K_+(s+V_+)}} & , T^{x_j}_{-+} = \dfrac{2\sqrt{K_-(s+V_-)}}{\sqrt{K_-(s+V_-)}+\sqrt{K_+(s+V_+)}} \\ R^{x_j}_{++} = \dfrac{\sqrt{K_+(s+V_+)}-\sqrt{K_-(s+V_-)}}{\sqrt{K_-(s+V_-)}+\sqrt{K_+(s+V_+)}} & , R^{x_j}_{--} = \dfrac{\sqrt{K_-(s+V_-)}-\sqrt{K_+(s+V_+)}}{\sqrt{K_-(s+V_-)}+\sqrt{K_+(s+V_+)}} \\ K_- \equiv K(x_j-), K_+ \equiv K(x_j+) & , V_- \equiv V(x_j-), V_+ \equiv V(x_j+) \end{cases} \tag{4.1}$$

Definition 4.1. We define $\Omega_{y,x}$ as the set of path that connecting y and x, that is,

$$\Omega_{y,x} \equiv \{\gamma : \gamma : [0,1] \to \mathbb{R},\ \gamma \text{ is continuous with } \gamma(0) = y,\ \gamma(1) = x\}$$

Definition 4.2. Given $\gamma \in \Omega_{y,x}$, we define

$$d_i = \begin{cases} T^{x_j}_{+-} & , \text{ if } \gamma \text{ passes } x_j \text{ from left,} \\ T^{x_j}_{-+} & , \text{ if } \gamma \text{ passes } x_j \text{ from right,} \\ R^{x_j}_{++} & , \text{ if } \gamma \text{ passes } x_j \text{ from left,} \\ R^{x_j}_{--} & , \text{ if } \gamma \text{ passes } x_j \text{ from right.} \end{cases}$$

and $D(\gamma) \equiv \{d_1, d_2 \ldots\}$. Based on this, we define

$$m(\gamma) \equiv \begin{cases} 1 & , \text{ if } D(\gamma) = \varnothing, \\ \displaystyle\prod_{k=1}^{|D(\gamma)|} d_k & , \text{ otherwise .} \end{cases} \tag{4.2}$$

Theorem 4.3.

$$\mathcal{L}u = \sum_{\gamma \in \Omega_{y,x}} m(\gamma) \frac{e^{-\int_\gamma \sqrt{\frac{s+V(\gamma)}{K(\gamma)}}|d\gamma|}}{2\sqrt{K(y)(s+V(y))}} \tag{4.3}$$

5 Inverse of $m(\gamma)$

5.1 Inverse of $R_{\pm\pm}$

Proposition 5.1. The functions

$$f(z) = -\frac{\sqrt{1+z} - \sqrt{\frac{K_-}{K_+}}}{\sqrt{1+z} + \sqrt{\frac{K_-}{K_+}}}$$
$$g(z) = \frac{\sqrt{1+z} - \sqrt{\frac{K_+}{K_-}}}{\sqrt{1+z} + \sqrt{\frac{K_+}{K_-}}}$$

are analytic around $z = 0$ with

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n z_n, \qquad \text{for } |z| \le \frac{1}{2} \\ g(z) &= \sum_{n=0}^{\infty} b_n z_n, \qquad \text{for } |z| \le \frac{1}{2} \\ |a_n| &\le 1, \qquad |b_n| \le 1 \end{aligned} \tag{5.1}$$

and

$$a_0, b_0 = \frac{K_- - K_+}{(\sqrt{K_-} + \sqrt{K_+})^2} \tag{5.2}$$

Proof: First, it is clear that $f(z)$ is analytic on the domain $|z| < 1$. By Cauchy integral formula, there exists a_n such that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \text{for } |z| \le \frac{1}{2}$$

Now, for any $1 > \epsilon > 0$, we consider the expansion of f around 0 on the domain $|z| \le 1 - \epsilon$, we know by identity theorem that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad \text{for } |z| \le 1 - \epsilon$$

and

$$|a_n| = \left| \frac{1}{2\pi i} \oint_{|z|=1-\epsilon} \frac{f(z)}{z^{n+1}} dz \right| \le \frac{1}{(1-\epsilon)^{n+1}}$$

since $|f(z)| \le 1$ for $|z| < 1$. As $\epsilon \to 0^+$, we can see that $|a_n| \le 1$. On the other hand, we know that

$$a_0 = f(0) = \frac{K_- - K_+}{(\sqrt{K_-} + \sqrt{K_+})^2}.$$

Similar way gives us the result of g. □

Proposition 5.2. Suppose $|V_+ - V_-| \ll 1$ and $V_-, V_+ \geq 0$. The function

$$R_{--}(s) = \frac{\sqrt{K_-(s+V_-)} - \sqrt{K_+(s+V_+)}}{\sqrt{K_-(s+V_-)} + \sqrt{K_+(s+V_+)}}$$

has the form

$$R_{--}(s) = \sum_{n=0}^{\infty} \left[a_n \left(\frac{V_+ - V_-}{s+V_-} \right)^n \right] = \sum_{n=0}^{\infty} \left[b_n \left(\frac{V_- - V_+}{s+V_+} \right)^n \right] \tag{5.3}$$

Proof: At the first, we know that

$$\begin{aligned} R_{--}(s) &= \frac{\sqrt{K_-(s+V_-)} - \sqrt{K_+(s+V_+)}}{\sqrt{K_-(s+V_-)} + \sqrt{K_+(s+V_+)}} \\ &= -\frac{\sqrt{1 + \frac{V_+ - V_-}{s+V_-}} - \sqrt{\frac{K_-}{K_+}}}{\sqrt{1 + \frac{V_+ - V_-}{s+V_-}} + \sqrt{\frac{K_-}{K_+}}} \end{aligned}$$

Since $|V_+ - V_-| \ll 1$ and $V_-, V_+ \geq 0$, we have $|\frac{V_+ - V_-}{s+V_-}| \leq \frac{1}{2}$. By (5.1), we see that

$$R_{--}(s) = \sum_{n=0}^{\infty} a_n \left(\frac{V_+ - V_-}{s+V_-} \right)^n$$

which is our desired form. On the other hand,

$$R_{--}(s) = \frac{\sqrt{1 + \frac{V_- - V_+}{s+V_+}} - \sqrt{\frac{K_+}{K_-}}}{\sqrt{1 + \frac{V_- - V_+}{s+V_+}} + \sqrt{\frac{K_+}{K_-}}}$$

Since $|V_+ - V_-| \ll 1$ and $V_-, V_+ \geq 0$, we have $|\frac{V_- - V_+}{s+V_+}| \leq \frac{1}{2}$. By (5.1), we see that

$$R_{--}(s) = \sum_{n=0}^{\infty} b_n \left(\frac{V_- - V_+}{s+V_+} \right)^n$$

□

Proposition 5.3. Suppose $|V_+ - V_-| \ll 1$ and $V_-, V_+ \geq 0$. Then

$$\begin{aligned} \mathcal{L}^{-1}(R_{--}) &= \frac{K_- - K_+}{(\sqrt{K_-} + \sqrt{K_+})^2} \delta(t) + \sum_{n=1}^{\infty} a_n e^{-V_- t} (V_+ - V_-)^n \frac{t^{n-1}}{(n-1)!} \\ &= \frac{K_- - K_+}{(\sqrt{K_-} + \sqrt{K_+})^2} \delta(t) + \sum_{n=1}^{\infty} b_n e^{-V_+ t} (V_- - V_+)^n \frac{t^{n-1}}{(n-1)!} \end{aligned} \tag{5.4}$$

Proof: By (5.3), we have

$$\begin{aligned} R_{--}(s) &= \sum_{n=0}^{\infty} a_n \left(\frac{V_+ - V_-}{s + V_-}\right)^n \\ &= a_0 + \sum_{n=1}^{\infty} a_n \left(\frac{V_+ - V_-}{s + V_-}\right)^n \\ &= \frac{K_- - K_+}{(\sqrt{K_-} + \sqrt{K_+})^2} + \sum_{n=1}^{\infty} a_n \left(\frac{V_+ - V_-}{s + V_-}\right)^n \end{aligned}$$

Hence,

$$\begin{aligned} \mathcal{L}^{-1}(R_{--}) &= \mathcal{L}^{-1}\left(\frac{K_- - K_+}{(\sqrt{K_-} + \sqrt{K_+})^2}\right) + \sum_{n=1}^{\infty} a_n \mathcal{L}^{-1}\left(\frac{V_+ - V_-}{s + V_-}\right)^n \\ &= \frac{K_- - K_+}{(\sqrt{K_-} + \sqrt{K_+})^2}\delta(t) + \sum_{n=1}^{\infty} a_n e^{-V_- t}(V_+ - V_-)^n \frac{t^{n-1}}{(n-1)!} \end{aligned}$$

Similar way gives us

$$\mathcal{L}^{-1}(R_{--}) = \frac{K_- - K_+}{(\sqrt{K_-} + \sqrt{K_+})^2}\delta(t) + \sum_{n=1}^{\infty} b_n e^{-V_+ t}(V_- - V_+)^n \frac{t^{n-1}}{(n-1)!}$$

□

Corollary 5.4.

$$\left|\mathcal{L}^{-1}(R_{--} - \frac{K_- - K_+}{(\sqrt{K_-} + \sqrt{K_+})^2})\right| \leq |V_+ - V_-| e^{-\min(V_-, V_+)t} \tag{5.5}$$

Proof: (5.4) tells us that,

$$\mathcal{L}^{-1}(R_{--} - \frac{K_- - K_+}{(\sqrt{K_-} + \sqrt{K_+})^2}) = \sum_{n=1}^{\infty} a_n e^{-V_- t}(V_+ - V_-)^n \frac{t^{n-1}}{(n-1)!}$$

Hence,

$$\begin{aligned} \left|\mathcal{L}^{-1}(R_{--} - \frac{K_- - K_+}{(\sqrt{K_-} + \sqrt{K_+})^2})\right| &\leq \sum_{n=1}^{\infty} \left| a_n e^{-V_- t}(V_+ - V_-)^n \frac{t^{n-1}}{(n-1)!}\right| \\ &\leq |V_+ - V_-| e^{-V_- t} \sum_{n=1}^{\infty} |a_n| \frac{(|V_+ - V_-|t)^{n-1}}{(n-1)!} \end{aligned}$$

By (5.1), we have $|a_n| \leq 1$, so,

$$\begin{aligned} \left|\mathcal{L}^{-1}(R_{--} - \frac{K_- - K_+}{(\sqrt{K_-} + \sqrt{K_+})^2})\right| &\leq |V_+ - V_-| e^{-V_- t} \sum_{n=1}^{\infty} \frac{(|V_+ - V_-|t)^{n-1}}{(n-1)!} \\ &\leq |V_+ - V_-| e^{-V_- t + |V_+ - V_-|t} \end{aligned} \tag{5.6}$$

Another expansion of R_{--} given by (5.4) gives us

$$\left|\mathcal{L}^{-1}(R_{--} - \frac{K_- - K_+}{(\sqrt{K_-} + \sqrt{K_+})^2})\right| \leq |V_+ - V_-| e^{-V_+ t + |V_+ - V_-| t} \tag{5.7}$$

(5.6) and (5.7) tells us that

$$\left|\mathcal{L}^{-1}(R_{--} - \frac{K_- - K_+}{(\sqrt{K_-} + \sqrt{K_+})^2})\right| \leq |V_+ - V_-| e^{-\min(V_-, V_+) t}$$

□

Corollary 5.5.

$$\left|\mathcal{L}^{-1}(R_{++} - \frac{K_+ - K_-}{(\sqrt{K_-} + \sqrt{K_+})^2})\right| \leq |V_+ - V_-| e^{-\min(V_-, V_+) t} \tag{5.8}$$

Proof: Since $R_{++} = -R_{--}$, this result followed by (5.5). □

Corollary 5.6. For any $x_j \in J$, we have

$$\left|\mathcal{L}^{-1}(R^{x_j}_{\pm\pm} - \frac{K_+ - K_-}{(\sqrt{K_-} + \sqrt{K_+})^2})\right| \leq (\|V\|_{\mathrm{BV}}) e^{-\inf(V) t} \tag{5.9}$$

Proof: Directly from Corollary 4.5. □

Corollary 5.7. Given any n reflection coefficient R_j, $j = 1, \ldots, n$ (they can be either reflect from left or right), with the form given by (5.3) that

$$R_j(s) = c_j + \sum_{n_j=1}^{\infty} \tau_{n_j} \left(\frac{V^j_- - V^j_+}{s + V^j_+} \right)^{n_j}$$

where

$$c_j = \begin{cases} \dfrac{K^j_- - K^j_+}{(\sqrt{K^j_-} + \sqrt{K^j_+})^2}, & \text{if } R_j \text{ is reflection coefficient from the left,} \\ \dfrac{K^j_+ - K^j_-}{(\sqrt{K^j_-} + \sqrt{K^j_+})^2}, & \text{if } R_j \text{ is reflection coefficient from the right} \end{cases}$$

then

$$\left|\mathcal{L}^{-1}(\prod_{j=1}^{n} R_j - \prod_{j=1}^{n} c_j)\right| \leq e^{-\min_j (V^j_-, V^j_+) t} \sum_{k=1}^{n} \binom{n}{k} (\max_j c_j)^{n-k} (\max_j |V^j_- - V^j_+|)^k \frac{t^{k-1}}{(k-1)!} \tag{5.10}$$

Proof: We prove it by induction. It is clear that by (5.5) and (5.8) that, the case $n = 1$ holds. Assume that the inequality holds for $n = m$. Now, we consider $n = m + 1$, we have

$$\begin{aligned}
\prod_{j=1}^{m+1} R_j - \prod_{j=1}^{m+1} c_j &= (R_{m+1} - c_{m+1} + c_{m+1}) \left(\prod_{j=1}^{m} R_j - \prod_{j=1}^{m} c_j + \prod_{j=1}^{m} c_j \right) - \prod_{j=1}^{m+1} c_j \\
&= (R_{m+1} - c_{m+1}) \left(\prod_{j=1}^{m} R_j - \prod_{j=1}^{m} c_j \right) + c_{m+1} \left(\prod_{j=1}^{m} R_j - \prod_{j=1}^{m} c_j \right) + (R_{m+1} - c_{m+1}) \prod_{j=1}^{m} c_j
\end{aligned}$$

So,

$$
\begin{aligned}
&\left|\mathcal{L}^{-1}(\prod_{j=1}^{m+1} R_j - \prod_{j=1}^{m+1} c_j)\right| \\
&= \left|\mathcal{L}^{-1}(R_{m+1}-c_{m+1}) *_t \mathcal{L}^{-1}\left(\prod_{j=1}^{m} R_j - \prod_{j=1}^{m} c_j\right) + c_{m+1}\mathcal{L}^{-1}\left(\prod_{j=1}^{m} R_j - \prod_{j=1}^{m} c_j\right) + \prod_{j=1}^{m} c_j \mathcal{L}^{-1}(R_{m+1}-c_{m+1})\right| \\
&\leq \left|\mathcal{L}^{-1}(R_{m+1}-c_{m+1}) *_t \mathcal{L}^{-1}\left(\prod_{j=1}^{m} R_j - \prod_{j=1}^{m} c_j\right)\right| + |c_{m+1}|\left|\mathcal{L}^{-1}\left(\prod_{j=1}^{m} R_j - \prod_{j=1}^{m} c_j\right)\right| \\
&\qquad + \left|\prod_{j=1}^{m} c_j\right|\left|\mathcal{L}^{-1}(R_{m+1}-c_{m+1})\right| \\
&\leq \int_0^t e^{-\min_j(V_-^j,V_+^j)\zeta} \sum_{k=1}^{m} \binom{m}{k} (\max_j c_j)^{m-k} (\max_j |V_-^j - V_+^j|)^k \frac{\zeta^{k-1}}{(k-1)!} \cdot |V_-^{m+1} - V_+^{m+1}| e^{-\min_j(V_-^j,V_+^j)(t-\zeta)} d\zeta \\
&\qquad + (\max_j c_j) e^{-\min_j(V_-^j,V_+^j)t} \sum_{k=1}^{m} \binom{m}{k} (\max_j c_j)^{m-k} (\max_j |V_-^j - V_+^j|)^k \frac{t^{k-1}}{(k-1)!} \\
&\qquad + (\max_j c_j)^m |V_-^{m+1} - V_+^{m+1}| e^{-\min_j(V_-^j,V_+^j)t} \\
&\leq e^{-\min_j(V_-^j,V_+^j)t} \sum_{k=1}^{m} \binom{m}{k} (\max_j c_j)^{m-k} (\max_j |V_-^j - V_+^j|)^{k+1} \frac{t^k}{k!} \\
&\quad + e^{-\min_j(V_-^j,V_+^j)t} \sum_{k=1}^{m} \binom{m}{k} (\max_j c_j)^{m+1-k} (\max_j |V_-^j - V_+^j|)^k \frac{t^{k-1}}{(k-1)!} \\
&\quad + e^{-\min_j(V_-^j,V_+^j)t} (\max_j c_j)^m (\max_j |V_-^j - V_+^j|) \\
&\leq e^{-\min_j(V_-^j,V_+^j)t} \sum_{k=2}^{m+1} \binom{m}{k-1} (\max_j c_j)^{m+1-k} (\max_j |V_-^j - V_+^j|)^k \frac{t^{k-1}}{(k-1)!} \\
&\quad + e^{-\min_j(V_-^j,V_+^j)t} \sum_{k=1}^{m} \binom{m}{k} (\max_j c_j)^{m+1-k} (\max_j |V_-^j - V_+^j|)^k \frac{t^{k-1}}{(k-1)!} \\
&\quad + e^{-\min_j(V_-^j,V_+^j)t} (\max_j c_j)^m (\max_j |V_-^j - V_+^j|) \\
&\leq e^{-\min_j(V_-^j,V_+^j)t} \sum_{k=2}^{m} \binom{m}{k-1} (\max_j c_j)^{m+1-k} (\max_j |V_-^j - V_+^j|)^k \frac{t^{k-1}}{(k-1)!} \\
&\quad + e^{-\min_j(V_-^j,V_+^j)t} \sum_{k=2}^{m} \binom{m}{k} (\max_j c_j)^{m+1-k} (\max_j |V_-^j - V_+^j|)^k \frac{t^{k-1}}{(k-1)!} \\
&\quad + (m+1) e^{-\min_j(V_-^j,V_+^j)t} (\max_j c_j)^m (\max_j |V_-^j - V_+^j|) + (\max_j |V_-^j - V_+^j|)^{m+1} \frac{t^m}{m!} \\
&\leq e^{-\min_j(V_-^j,V_+^j)t} \sum_{k=1}^{m+1} \binom{m+1}{k} (\max_j c_j)^{n-k} (\max_j |V_-^j - V_+^j|)^k \frac{t^{k-1}}{(k-1)!}
\end{aligned}
$$

□

Corollary 5.8.

$$\left|\mathcal{L}^{-1}(\prod_{j=1}^{n} R_j - \prod_{j=1}^{n} c_j)\right| \leq \frac{(\max_j |V_-^j - V_+^j|) e^{-\min_j (V_-^j, V_+^j) t} e^{\frac{\max_j |V_-^j - V_+^j| t}{\max_j c_j}}}{\max_j c_j} \left[(2 \max_j c_j)^n \right] \tag{5.11}$$

Proof:

$$\left|\mathcal{L}^{-1}(\prod_{j=1}^{n} R_j - \prod_{j=1}^{n} c_j)\right|$$

$$\leq e^{-\min_j (V_-^j, V_+^j) t} \sum_{k=1}^{n} \binom{n}{k} (\max_j c_j)^{n-k} (\max_j |V_-^j - V_+^j|)^k \frac{t^{k-1}}{(k-1)!}$$

$$\leq (\max_j |V_-^j - V_+^j|) e^{-\min_j (V_-^j, V_+^j) t} \sum_{k=1}^{n} \binom{n}{k} (\max_j c_j)^{n-k} \frac{(\max_j |V_-^j - V_+^j| t)^{k-1}}{(k-1)!}$$

$$\leq \frac{(\max_j |V_-^j - V_+^j|) e^{-\min_j (V_-^j, V_+^j) t}}{\max_j c_j} \sum_{k=1}^{n} \binom{n}{k} (\max_j c_j)^n \frac{1}{(k-1)!} \left(\frac{\max_j |V_-^j - V_+^j| t}{\max_j c_j} \right)^{k-1}$$

$$\leq \frac{(\max_j |V_-^j - V_+^j|) e^{-\min_j (V_-^j, V_+^j) t}}{\max_j c_j} \sum_{k=1}^{n} (2 \max_j c_j)^n \frac{1}{(k-1)!} \left(\frac{\max_j |V_-^j - V_+^j| t}{\max_j c_j} \right)^{k-1}$$

$$\leq \frac{(\max_j |V_-^j - V_+^j|) e^{-\min_j (V_-^j, V_+^j) t} e^{\frac{\max_j |V_-^j - V_+^j| t}{\max_j c_j}}}{\max_j c_j} \left[(2 \max_j c_j)^n \right]$$

□

6 Estimation of Gaussian

Proposition 6.1. Given a sequence of function $f_1(t), f_2(t), \dots, f_n(t)$ with $f_k(t) \geq 0$ for all k and constants $c_1, c_2, \dots, c_n$ with $c_k > 0$ for all k. Then,

$$\left|(e^{-c_1 t} f_1(t)) *_t (e^{-c_2 t} f_2(t)) *_t \cdots *_t (e^{-c_n t} f_n(t))\right| \leq e^{-(\min_i c_i)t}(f_1(t) *_t f_2(t) *_t \cdots *_t f_n(t)) \tag{6.1}$$

Proof: We show it by induction. For $n = 1$, the inequality is clear. Suppose for $n = m$, the inequality holds. Now for $n = m + 1$, we have

$$\begin{aligned}
&\left|(e^{-c_1 t} f_1(t)) *_t (e^{-c_2 t} f_2(t)) *_t \cdots *_t (e^{-c_n t} f_n(t))\right| \\
&= (e^{-c_1 t} f_1(t)) *_t (e^{-c_2 t} f_2(t)) *_t \cdots *_t (e^{-c_n t} f_n(t)) \\
&= \left((e^{-c_1 t} f_1(t)) *_t \cdots *_t (e^{-c_m t} f_m(t))\right) *_t (e^{-c_{m+1} t} f_{m+1}(t)) \\
&= \int_0^t \left[(e^{-c_1 t} f_1(t)) *_t \cdots *_t (e^{-c_m t} f_m(t))\right](\tau)(e^{-c_{m+1}(t-\tau)} f_{m+1}(t-\tau))d\tau \\
&\leq \int_0^t e^{-(\min_{i=1,\dots,m} c_i)t}\left[(f_1(t) *_t f_2(t) *_t \cdots *_t f_m(t))\right](\tau)(e^{-c_{m+1}(t-\tau)} f_{m+1}(t-\tau))d\tau \\
&\leq e^{-(\min_{i=1,\dots,m,m+1} c_i)t} \int_0^t \left[(f_1(t) *_t f_2(t) *_t \cdots *_t f_m(t))\right](\tau) f_{m+1}(t-\tau)d\tau \\
&\leq e^{-(\min_{i=1,\dots,m,m+1} c_i)t}(f_1(t) *_t f_2(t) *_t \cdots *_t f_n(t))
\end{aligned}$$

□

Proposition 6.2.

$$\left|\mathcal{L}^{-1}\left(\frac{e^{-\int_\gamma\sqrt{\frac{s+V(\gamma)}{K(\gamma)}}|d\gamma|}}{2\sqrt{K(y)(s+V(y))}}\right)\right| \le e^{-(\inf V)t}\frac{e^{-\frac{1}{4t}(\int\frac{d\gamma}{\sqrt{K(\gamma)}})^2}}{2\sqrt{K(y)\pi t}} \tag{6.2}$$

Proof: Suppose $\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_M$ with $K(\gamma_1) = K(y)$, $V(\gamma_1) = V(y)$ and $K(\gamma_k) \equiv K_k$, $V(\gamma_k) \equiv V_k$, $k = 2, \dots, M$ for some constant K_k and V_k, then by (6.1) and the fact that

$$\mathcal{L}^{-1}\left(\frac{e^{-\int_{\gamma_1}\sqrt{\frac{s}{K(y)}}|d\gamma|}}{2\sqrt{sK(y)}}\right) \ge 0, \quad \mathcal{L}^{-1}\left(e^{-\int_{\gamma_k}\sqrt{\frac{s}{K_k}}|d\gamma|}\right) \ge 0, \quad \forall k = 2, \dots, M$$

we have

$$\begin{aligned}
\left|\mathcal{L}^{-1}\left(\frac{e^{-\int_\gamma\sqrt{\frac{s+V(\gamma)}{K(\gamma)}}|d\gamma|}}{2\sqrt{K(y)(s+V(y))}}\right)\right| &= \left|\mathcal{L}^{-1}\left(\frac{e^{-\int_{\gamma_1}\sqrt{\frac{s+V(y)}{K(y)}}|d\gamma|-\sum_{k=2}^M\int_{\gamma_k}\sqrt{\frac{s+V_k}{K_k}}|d\gamma|}}{2\sqrt{K(y)(s+V(y))}}\right)\right| \\
&= \left|\mathcal{L}^{-1}\left(\frac{e^{-\int_{\gamma_1}\sqrt{\frac{s+V(y)}{K(y)}}|d\gamma|}}{2\sqrt{K(y)(s+V(y))}}\right) *_t \mathcal{L}^{-1}\left(e^{-\int_{\gamma_2}\sqrt{\frac{s+V_2}{K_2}}|d\gamma|}\right) *_t \cdots *_t \right. \\
&\qquad \left. \mathcal{L}^{-1}\left(e^{-\int_{\gamma_M}\sqrt{\frac{s+V_M}{K_M}}|d\gamma|}\right)\right| \\
&\le \left|\left[e^{-V(y)t}\mathcal{L}^{-1}\left(\frac{e^{-\int_{\gamma_1}\sqrt{\frac{s}{K(y)}}|d\gamma|}}{2\sqrt{sK(y)}}\right)\right] *_t \left[e^{-V_2 t}\mathcal{L}^{-1}\left(e^{-\int_{\gamma_2}\sqrt{\frac{s}{K_2}}|d\gamma|}\right)\right] *_t \cdots *_t \right. \\
&\qquad \left. \left[e^{-V_M t}\mathcal{L}^{-1}\left(e^{-\int_{\gamma_M}\sqrt{\frac{s}{K_M}}|d\gamma|}\right)\right]\right| \\
&\le e^{-(\inf V)t}\mathcal{L}^{-1}\left(\frac{e^{-\int_{\gamma_1}\sqrt{\frac{s}{K(y)}}|d\gamma|}}{2\sqrt{sK(y)}}\right) *_t \mathcal{L}^{-1}\left(e^{-\int_{\gamma_2}\sqrt{\frac{s}{K_2}}|d\gamma|}\right) *_t \cdots *_t \\
&\qquad \mathcal{L}^{-1}\left(e^{-\int_{\gamma_M}\sqrt{\frac{s}{K_M}}|d\gamma|}\right) \\
&\le e^{-(\inf V)t}\mathcal{L}^{-1}\left(\frac{e^{-\int_{\gamma}\sqrt{\frac{s}{K(\gamma)}}|d\gamma|}}{2\sqrt{sK(y)}}\right) \\
&\le e^{-(\inf V)t}\frac{e^{-\frac{1}{4t}(\int\frac{d\gamma}{\sqrt{K(\gamma)}})^2}}{2\sqrt{K(y)\pi t}}
\end{aligned}$$

Here, we have used proposition 5.1. □

7 Example of two jump

Assume that $J = \{x_1, x_2\}$ with $x_1 < x_2$. Moreover, suppose that

$$K(x) = \begin{cases} K_1, & \text{if } x < x_1, \\ K_2, & \text{if } x_1 < x < x_2, \\ K_3, & \text{if } x > x_2, \end{cases} \tag{7.1}$$

and

$$V(x) = \begin{cases} V_1, & \text{if } x < x_1, \\ V_2, & \text{if } x_1 < x < x_2, \\ V_3, & \text{if } x > x_2. \end{cases} \tag{7.2}$$

where $V_1, V_2, V_3 > 0$. We discuss the case $y < x_1$.

For $x < x_1$, we have

$\mathcal{L}u(x,s) =$

$$\frac{e^{-\sqrt{\frac{s+V_1}{K_1}}|x-y|}}{2\sqrt{K_1(s+V_1)}} + R^{x_1}_{--}\frac{e^{-\sqrt{\frac{s+V_1}{K_1}}(|y-x_1|+|x-x_1|)}}{2\sqrt{K_1(s+V_1)}} + \sum_{j=1}^{\infty}(T^{x_1}_{-+})(T^{x_1}_{+-})(R^{x_2}_{--})^{2j-1}\frac{e^{-\sqrt{\frac{s+V_1}{K_1}}(|y-x_1|+|x-x_1|)-\sqrt{\frac{s+V_2}{K_2}}(2j|x_1-x_2|)}}{2\sqrt{K_1(s+V_1)}} \tag{7.3}$$

For $x_1 < x < x_2$, we have

$$\mathcal{L}u(x,s) = \sum_{j=0}^{\infty}(T^{x_1}_{-+})(R^{x_2}_{--})^{j}(R^{x_1}_{++})^{j}\frac{e^{-\sqrt{\frac{s+V_1}{K_1}}|y-x_1|-\sqrt{\frac{s+V_2}{K_2}}(|x-x_1|+2j|x_1-x_2|)}}{2\sqrt{K_1(s+V_1)}} \tag{7.4}$$

$$+\sum_{j=0}^{\infty}(T^{x_1}_{-+})(R^{x_2}_{--})^{j+1}(R^{x_1}_{++})^{j}\frac{e^{-\sqrt{\frac{s+V_1}{K_1}}|y-x_1|-\sqrt{\frac{s+V_2}{K_2}}(|x-x_1|+(2j+1)|x_1-x_2|)}}{2\sqrt{K_1(s+V_1)}}$$

For $x > x_2$, we have

$$\mathcal{L}u(x,s) = \sum_{j=0}^{\infty}(T^{x_1}_{-+})(T^{x_2}_{-+})(R^{x_1}_{++})^{j}(R^{x_2}_{--})^{j}\frac{e^{-\sqrt{\frac{s+V_1}{K_1}}|y-x_1|-\sqrt{\frac{s+V_2}{K_2}}((2j+1)|x_1-x_2|)-\sqrt{\frac{s+V_3}{K_3}}|x-x_2|}}{2\sqrt{K_1(s+V_1)}} \tag{7.5}$$

Now, we define

$$\begin{aligned} c^{x_1}_{--} &= \frac{K_1-K_2}{(\sqrt{K_1}+\sqrt{K_2})^2}, \qquad & c^{x_1}_{++} &= \frac{K_2-K_1}{(\sqrt{K_1}+\sqrt{K_2})^2} \\ c^{x_2}_{--} &= \frac{K_2-K_3}{(\sqrt{K_2}+\sqrt{K_3})^2}, \qquad & c^{x_2}_{++} &= \frac{K_3-K_2}{(\sqrt{K_2}+\sqrt{K_3})^2} \end{aligned} \tag{7.6}$$

and $\alpha = \max\{|c^{x_1}_{--}|, |c^{x_1}_{++}|, |c^{x_2}_{--}|, |c^{x_2}_{++}|\}$.

Proposition 7.1. Suppose $\|V\|_{BV} \ll 1$ and $V_i > 0$ for $i = 1, 2, 3$. Given any number $l \in \mathbb{N}$, we have

$$\sum_{j=l}^{\infty}\left|\mathcal{L}^{-1}\left((T^{x_1}_{-+})(T^{x_1}_{+-})(R^{x_2}_{--})^{2j-1} - (1+c^{x_1}_{--})(1-c^{x_1}_{--})(c^{x_1}_{--})^2(c^{x_2}_{--})^{2j-1}\right)\right| \le \frac{2\|V\|_{BV}e^{-\min_i(V_i)t+\frac{t\|V\|_{BV}}{\alpha}}}{\alpha^2(1-4\alpha^2)}(2\alpha)^{2l} \tag{7.7}$$

Proof:

$$\sum_{j=l}^{\infty}\left|\mathcal{L}^{-1}\left((T^{x_1}_{-+})(T^{x_1}_{+-})(R^{x_2}_{--})^{2j-1}-(1+c^{x_1}_{--})(1-c^{x_1}_{--})(c^{x_1}_{--})^2(c^{x_2}_{--})^{2j-1}\right)\right|$$
$$=\sum_{j=l}^{\infty}\left|\mathcal{L}^{-1}\left((T^{x_1}_{-+})(T^{x_1}_{+-})(R^{x_2}_{--})^{2j-1}-\left((c^{x_2}_{--})^{2j-1}-(c^{x_1}_{--})^2(c^{x_2}_{--})^{2j-1}\right)\right)\right|$$
$$=\sum_{j=l}^{\infty}\left|\mathcal{L}^{-1}\left((1-R^{x_1}_{--})(1+R^{x_1}_{--})(R^{x_2}_{--})^{2j-1}-\left((c^{x_2}_{--})^{2j-1}-(c^{x_1}_{--})^2(c^{x_2}_{--})^{2j-1}\right)\right)\right|$$
$$=\sum_{j=l}^{\infty}\left|\mathcal{L}^{-1}\left((R^{x_2}_{--})^{2j-1}-(R^{x_1}_{--})^2(R^{x_2}_{--})^{2j-1}-\left((c^{x_2}_{--})^{2j-1}-(c^{x_1}_{--})^2(c^{x_2}_{--})^{2j-1}\right)\right)\right|$$
$$\leq\sum_{j=l}^{\infty}\left|\mathcal{L}^{-1}\left((R^{x_2}_{--})^{2j-1}-(c^{x_2}_{--})^{2j-1}\right)\right|+\sum_{j=l}^{\infty}\left|\mathcal{L}^{-1}\left((R^{x_1}_{--})^2(R^{x_2}_{--})^{2j-1}-(c^{x_1}_{--})^2(c^{x_2}_{--})^{2j-1}\right)\right|$$

By Corollary 4.8, we have

$$\sum_{j=l}^{\infty}\left|\mathcal{L}^{-1}\left((T^{x_1}_{-+})(T^{x_1}_{+-})(R^{x_2}_{--})^{2j-1}-(1+c^{x_1}_{--})(1-c^{x_1}_{--})(c^{x_1}_{--})^2(c^{x_2}_{--})^{2j-1}\right)\right|$$
$$\leq\frac{\|V\|_{BV}e^{-\min_i(V_i)t}e^{\frac{\|V\|_{BV}t}{\alpha}}}{\alpha}\sum_{j=l}^{\infty}\left((2\alpha)^{2j-1}+(2\alpha)^{2j+1}\right)$$
$$\leq\frac{2\|V\|_{BV}e^{-\min_i(V_i)t}e^{\frac{\|V\|_{BV}t}{\alpha}}}{\alpha}\sum_{j=l}^{\infty}\left((2\alpha)^{2j-1}\right)$$
$$\leq\frac{2\|V\|_{BV}e^{-\min_i(V_i)t+\frac{t\|V\|_{BV}}{\alpha}}}{\alpha^2(1-4\alpha^2)}(2\alpha)^{2l}$$

□

By similar calculation, we can conclude

Proposition 7.2. Suppose $\|V\|_{BV} \ll 1$ and $V_i > 0$ for $i = 1, 2, 3$. Given any number $l \in \mathbb{N}$, we have

$$\sum_{j=l}^{\infty} \left| \mathcal{L}^{-1}\left((T_{-+}^{x_1})(R_{--}^{x_2})^j (R_{++}^{x_1})^j - (1 + c_{--}^{x_1})(c_{--}^{x_2})^j (c_{--}^{x_1})^j \right) \right| \leq O(1)\|V\|_{BV} e^{-\min_i (V_i)t + \frac{t\|V\|_{BV}}{\alpha}} (2\alpha)^{2l} \tag{7.8}$$

and

$$\begin{aligned} &\sum_{j=l}^{\infty} \left| \mathcal{L}^{-1}\left((T_{-+}^{x_1})(R_{--}^{x_2})^{j+1} (R_{++}^{x_1})^j - (1 + c_{--}^{x_1})(c_{--}^{x_2})^{j+1} (c_{--}^{x_1})^j \right) \right| \\ &\leq O(1)\|V\|_{BV} e^{-\min_i (V_i)t + \frac{t\|V\|_{BV}}{\alpha}} (2\alpha)^{2l} \end{aligned} \tag{7.9}$$

and

$$\sum_{j=l}^{\infty} \left| \mathcal{L}^{-1}\left((T_{-+}^{x_1})(T_{-+}^{x_2})(R_{++}^{x_1})^j (R_{--}^{x_2})^j - (1 + c_{--}^{x_1})(1 + c_{--}^{x_2})(c_{++}^{x_1})^j (c_{--}^{x_2})^j) \right) \right| \tag{7.10}$$

$$\leq O(1)\|V\|_{BV} e^{-\min_i (V_i)t + \frac{t\|V\|_{BV}}{\alpha}} (2\alpha)^{2l} \tag{7.11}$$

Corollary 7.3. If $\alpha \ll 1$, u converges absolutely in all cases.

Proof: We show that for the case $x < x_1$, u converges absolutely, for other cases, the proof is similar. First, by (7.3), we know that

$$\mathcal{L}u(x,s) =$$
$$\frac{e^{-\sqrt{\frac{s+V_1}{K_1}}|x-y|}}{2\sqrt{K_1(s+V_1)}} + R_{--}^{x_1} \frac{e^{-\sqrt{\frac{s+V_1}{K_1}}(|y-x_1|+|x-x_1|)}}{2\sqrt{K_1(s+V_1)}} + \sum_{j=1}^{\infty} (T_{-+}^{x_1})(T_{+-}^{x_1})(R_{--}^{x_2})^{2j-1} \frac{e^{-\sqrt{\frac{s+V_1}{K_1}}(|y-x_1|+|x-x_1|) - \sqrt{\frac{s+V_2}{K_2}}(2j|x_1-x_2|)}}{2\sqrt{K_1(s+V_1)}}$$

So, we just need to show that

$$\sum_{j=1}^{\infty} \mathcal{L}^{-1}\left((T_{-+}^{x_1})(T_{+-}^{x_1})(R_{--}^{x_2})^{2j-1} \frac{e^{-\sqrt{\frac{s+V_1}{K_1}}(|y-x_1|+|x-x_1|) - \sqrt{\frac{s+V_2}{K_2}}(2j|x_1-x_2|)}}{2\sqrt{K_1(s+V_1)}} \right)$$

converges absolutely.

Let $C_j = (1+c^{x_1}_{--})(1-c^{x_1}_{--})(c^{x_1}_{--})^2(c^{x_2}_{--})^{2j-1}$. It is clear that $C_j > 0$. By Proposition 5.2,

$$\begin{aligned}
&\sum_{j=1}^{\infty}\left|\mathcal{L}^{-1}\left((T^{x_1}_{-+})(T^{x_1}_{+-})(R^{x_2}_{--})^{2j-1}\frac{e^{-\sqrt{\frac{s+V_1}{K_1}}(|y-x_1|+|x-x_1|)-\sqrt{\frac{s+V_2}{K_2}}(2j|x_1-x_2|)}}{2\sqrt{K_1(s+V_1)}}\right)\right| \\
&=\sum_{j=1}^{\infty}\left|\mathcal{L}^{-1}\left((T^{x_1}_{-+})(T^{x_1}_{+-})(R^{x_2}_{--})^{2j-1}-C_j\right)*_t\mathcal{L}^{-1}\left(\frac{e^{-\sqrt{\frac{s+V_1}{K_1}}(|y-x_1|+|x-x_1|)-\sqrt{\frac{s+V_2}{K_2}}(2j|x_1-x_2|)}}{2\sqrt{K_1(s+V_1)}}\right)\right. \\
&\quad\left.+C_j\mathcal{L}^{-1}\left(\frac{e^{-\sqrt{\frac{s+V_1}{K_1}}(|y-x_1|+|x-x_1|)-\sqrt{\frac{s+V_2}{K_2}}(2j|x_1-x_2|)}}{2\sqrt{K_1(s+V_1)}}\right)\right| \\
&\le\sum_{j=1}^{\infty}\int_0^t\left|\mathcal{L}^{-1}\left((T^{x_1}_{-+})(T^{x_1}_{+-})(R^{x_2}_{--})^{2j-1}-C_j\right)\right|(t-\tau)\left|\mathcal{L}^{-1}\left(\frac{e^{-\sqrt{\frac{s+V_1}{K_1}}(|y-x_1|+|x-x_1|)-\sqrt{\frac{s+V_2}{K_2}}(2j|x_1-x_2|)}}{2\sqrt{K_1(s+V_1)}}\right)\right|(\tau)d\tau \\
&\quad+\sum_{j=1}^{\infty}C_j\left|e^{-(\inf V)t}\frac{e^{-\frac{1}{4t}(\frac{|y-x_1|+|x-x_1|}{\sqrt{K_1}}+\frac{2j|x_1-x_2|}{\sqrt{K_2}})^2}}{2\sqrt{K_1\pi t}}\right| \\
&\le\int_0^t\sum_{j=1}^{\infty}\left|\mathcal{L}^{-1}\left((T^{x_1}_{-+})(T^{x_1}_{+-})(R^{x_2}_{--})^{2j-1}-C_j\right)\right|(t-\tau)\left|\mathcal{L}^{-1}\left(\frac{e^{-\sqrt{\frac{s+V_1}{K_1}}(|y-x_1|+|x-x_1|)-\sqrt{\frac{s+V_2}{K_2}}(2j|x_1-x_2|)}}{2\sqrt{K_1(s+V_1)}}\right)\right|(\tau)d\tau \\
&\quad+\sum_{j=1}^{\infty}C_j\left|e^{-(\inf V)t}\frac{e^{-\frac{1}{4t}(\frac{|y-x_1|+|x-x_1|}{\sqrt{K_1}}+\frac{2j|x_1-x_2|}{\sqrt{K_2}})^2}}{2\sqrt{K_1\pi t}}\right|
\end{aligned}$$

Since we know that $\sum_{j=1}^{\infty}C_j = \sum_{j=1}^{\infty}(1+c^{x_1}_{--})(1-c^{x_1}_{--})(c^{x_1}_{--})^2(c^{x_2}_{--})^{2j-1}$ is absolutely converges by the fact that $c^{x_2}_{--}\le\alpha\ll 1$ and we have Proposition 6.1, we can see that

$$\sum_{j=1}^{\infty}\mathcal{L}^{-1}\left((T^{x_1}_{-+})(T^{x_1}_{+-})(R^{x_2}_{--})^{2j-1}\frac{e^{-\sqrt{\frac{s+V_1}{K_1}}(|y-x_1|+|x-x_1|)-\sqrt{\frac{s+V_2}{K_2}}(2j|x_1-x_2|)}}{2\sqrt{K_1(s+V_1)}}\right)$$

is absolutely converges and,

$$\begin{aligned}
&\left|\sum_{j=1}^{\infty}\mathcal{L}^{-1}\left((T^{x_1}_{-+})(T^{x_1}_{+-})(R^{x_2}_{--})^{2j-1}\frac{e^{-\sqrt{\frac{s+V_1}{K_1}}(|y-x_1|+|x-x_1|)-\sqrt{\frac{s+V_2}{K_2}}(2j|x_1-x_2|)}}{2\sqrt{K_1(s+V_1)}}\right)\right| \\
&\le M_0\int_0^t\sum_{j=1}^{\infty}\left|\mathcal{L}^{-1}\left((T^{x_1}_{-+})(T^{x_1}_{+-})(R^{x_2}_{--})^{2j-1}-C_j\right)\right|dt+\sum_{j=1}^{\infty}C_je^{-(\inf V)t}\frac{e^{-\frac{1}{4t}(\frac{|y-x_1|+|x-x_1|}{\sqrt{K_1}})^2}}{2\sqrt{K_1\pi t}} \\
&\le M_0\int_0^t\frac{2\|V\|_{BV}e^{-\min_i(V_i)t+\frac{t\|V\|_{BV}}{\alpha}}}{\alpha^2(1-4\alpha^2)}(2\alpha)^2dt+e^{-(\inf V)t}\frac{e^{-\frac{1}{4t}(\frac{|y-x_1|+|x-x_1|}{\sqrt{K_1}})^2}}{2\sqrt{K_1\pi t}}\sum_{j=1}^{\infty}C_j
\end{aligned}$$

□